ACADEMIC PRESS

Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Sound and Vibration 266 (2003) 93–106

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

A new eigensolution of structures via dynamic condensation

Rongming Lin*, Yong Xia

School of Mechanical and Production Engineering, Nanyang Technological University, Nanyang Avenue, Singapore 639798, Singapore

Received 7 March 2002; accepted 3 September 2002

Abstract

Dynamic condensation (or model reduction) is a commonly used algorithm to fast estimate some low eigenvalues and corresponding eigenvectors of structures by reducing the order of the original structural model to a smaller one. This paper proposes a new eigensolution technique via iterated dynamic condensation. The technique retains all the inertia terms associated with the removed degrees of freedom in an iterated form, which generates the reduced mass matrix similar to that obtained in the Guyan reduction method with a frequency-dependent perturbed term. The corresponding eigenvalues and eigenvectors of interest are obtained as those of the Guyan reduction method with perturbations by using an eigensensitivity-based iterative method. The effectiveness and accuracy of the proposed technique are numerically verified by using a steel frame and the GARTEUR structure.

1. Introduction

Generalized eigenvalue problem is very important and fundamental in structural dynamics. It has been widely used in structural vibration analysis, finite element (FE) model updating, damage identification, etc. Due to the ever-increasing demand on the prediction accuracy of structural dynamic characteristics, the number of degrees of freedom (d.o.f.) in FE analysis becomes increasingly larger and examples involving tens of thousands of d.o.f.'s are not unusual in practice. This has led to the development of new methods so that the eigenvalues and corresponding eigenvectors of large structures can be solved accurately and effectively. In practice, for a large structure, only the first few lower modes are usually of practical interest, and

*Corresponding author. Tel.: 65-6790-4728; fax: 65-6791-1975.
E-mail addresses: mrmlin@ntu.edu.sg (R. Lin), myxia@ntu.edu.sg (Y. Xia).

doi:10.1016/S0022-460X(02)01278-6

solving the complete eigenproblem is not necessary and computationally very time consuming. In this case, condensation techniques are commonly used to give fast computation of the lowest eigenvalues and associated eigenvectors. This strategy removes some d.o.f (slave d.o.f.) of the original FE model and retains a much smaller set of d.o.f. (master d.o.f.'s), then solves the eigenfunction of the reduced model and approximates the eigensolutions of the original model.

One of the oldest and widely used condensation methods is Guyan reduction [1]. It is a static condensation method, which neglects the inertia terms of the slave d.o.f. Guyan reduction is only accurate at zero frequency. It is observed that the frequencies obtained by Guyan reduction are normally satisfactory in the domain of [0, $0.3f_s$], where f_s is the smallest eigenfrequency of the structure with all the master d.o.f. grounded [2,3]. In other higher domains, the errors of the results are large and sometimes unacceptable. To extend the domain and the validity of this technique, the master d.o.f. must be selected very carefully to increase the value of f_s. This has been studied by some researchers [4,5].

Contrast to static condensation, dynamic condensation methods consider the effects of the inertia terms of the slave d.o.f. Because the inertia terms are associated with the inverse of the dynamic stiffness matrix, they cannot be obtained directly. Several methods have been developed to estimate the effect from these inertia terms. One dynamic reduction was proposed by Paz [6,7], which considers the inertia terms at a given frequency shift in the transformation matrix between the reduced model and the original one. This technique is accurate at the perturbed frequency, rather than zero frequency as in Guyan reduction. Another dynamic reduction, improved reduced system (IRS) proposed by O'Callahan [8] includes a higher order term in Taylor series expansion of the inverse of the dynamic stiffness matrix. The method improves the accuracy of results and validity domain of the frequencies [8,9]. Friswell et al. developed an iterated IRS and proved its convergence [10,11]. Suarez and Singh [12], Qu and Fu [13] and Kim and Kang [14] presented similar techniques but with different iterative formulae. Leung [15] proposed a dynamic reduction technique by including some modes of the slavery substructure.

This paper aims to solve the eigenproblem of structures via a new iterative dynamic condensation technique. It develops the dynamic reduction method by including all the inertia terms in the transformation matrix without any approximation. The inverse matrix of the dynamic stiffness is transformed into an iterative formula. Then the reduced eigenfunction is derived, which includes a frequency-dependent term in the mass matrix. The frequency-dependent term can be looked as a perturbation of the mass matrix in Guyan reduction method. Therefore, the eigensolutions of the reduced model are obtained as those of Guyan reduction plus perturbation terms with an iterative eigensensitivity-based technique, proposed by Lin and Lim [16]. The main advantage of the proposed technique is that it can converge fast to the true eigensolutions of the original system. Only the eigensolutions in the reduced order are required once, as in Guyan reduction, then the eigenvalues and associated eigenvectors of interest can be obtained by direct iterations. Therefore, heavy computation is avoided.

A two-level frame with 66 d.o.f.'s and the GARTEUR structure with 216 d.o.f.'s are applied to illustrate the present algorithm. The results show that the first few lower eigenvalues and eigenvectors of interest are obtained accurately and efficiently, as compared with the conventional Guyan reduction and IRS method.

2. Theory

In FE analysis, the free vibration of an undamped structure with N d.o.f. is described by the general eigenvalue problem as

$$(\mathbf{K} - \lambda_i \mathbf{M})\mathbf{\Phi}_i = \mathbf{0}, \quad i = 1, 2, \ldots, N, \tag{1}$$

where $\mathbf{K}$ is the $N \times N$ symmetric stiffness matrix, $\mathbf{M}$ is the $N \times N$ symmetric mass matrix, λ_i and $\mathbf{\Phi}_i$ are the ith eigenvalue and corresponding mass-normalized eigenvector, respectively. For convenience, we only consider one mode first and the subscript can be omitted. Although the order of the structure might be very large, only the first n ($n \ll N$, for example, $n = 10$) eigensolutions are usually of practical interest.

Partitioning the d.o.f.'s in Eq. (1) into the master d.o.f. (retained) and slave d.o.f. (removed), the matrices and vector are split into sub-matrices and vectors. Hence one has

$$\left(\begin{bmatrix} \mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \mathbf{K}_{ms}^{\mathrm{T}} & \mathbf{K}_{ss} \end{bmatrix} - \lambda \begin{bmatrix} \mathbf{M}_{mm} & \mathbf{M}_{ms} \\ \mathbf{M}_{ms}^{\mathrm{T}} & \mathbf{M}_{ss} \end{bmatrix} \right) \begin{Bmatrix} \mathbf{\Phi}_m \\ \mathbf{\Phi}_s \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix}. \tag{2}$$

The subscripts "m" and "s" represent the master and slave d.o.f.'s, respectively, and superscript "T" is the transpose of the matrix. Assuming the sizes are m and s with $m + s = N$ and let $n < m$. From the second set of the above equation, $\mathbf{\Phi}_s$ can be expressed in terms of $\mathbf{\Phi}_m$ as

$$\mathbf{\Phi}_s = -(\mathbf{K}_{ss} - \lambda \mathbf{M}_{ss})^{-1}(\mathbf{K}_{ms}^{\mathrm{T}} - \lambda \mathbf{M}_{ms}^{\mathrm{T}})\mathbf{\Phi}_m = \mathbf{t}\mathbf{\Phi}_m, \tag{3}$$

where $\mathbf{t}$ is the transformation matrix between $\mathbf{\Phi}_m$ and $\mathbf{\Phi}_s$. Then the transformation between the master d.o.f.'s and the complete set of d.o.f. is

$$\mathbf{\Phi} = \begin{Bmatrix} \mathbf{\Phi}_m \\ \mathbf{\Phi}_s \end{Bmatrix} = \begin{Bmatrix} \mathbf{I}_m \\ \mathbf{t} \end{Bmatrix} \mathbf{\Phi}_m = \mathbf{T}\mathbf{\Phi}_m, \tag{4}$$

where $\mathbf{I}_m$ is the unit matrix of order m.

Substituting Eq. (4) into Eq. (1) and pre-multiplying $\mathbf{T}^{\mathrm{T}}$, one can obtain a reduced eigenvalue problem of order m:

$$(\mathbf{K}_R - \lambda \mathbf{M}_R)\mathbf{\Phi}_m = \mathbf{0}, \tag{5}$$

where $\mathbf{K}_R = \mathbf{T}^{\mathrm{T}}\mathbf{K}\mathbf{T}$ and $\mathbf{M}_R = \mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{T}$ are the reduced stiffness and mass matrices. $\mathbf{T}$ takes the form of

$$\mathbf{T} = \begin{bmatrix} \mathbf{I}_m \\ \mathbf{t} \end{bmatrix}, \tag{6a}$$

$$\mathbf{t} = -(\mathbf{K}_{ss} - \lambda \mathbf{M}_{ss})^{-1}(\mathbf{K}_{ms}^{\mathrm{T}} - \lambda \mathbf{M}_{ms}^{\mathrm{T}}). \tag{6b}$$

It is clear that the matrices $\mathbf{K}_R$ and $\mathbf{M}_R$ in Eq. (5) are frequency dependent and the eigenvalue problem cannot be directly solved by usual methods, such as Jacobi method etc. The commonly used Guyan reduction neglects the inertia terms in Eq. (6b) and yields in

$$\mathbf{T}_G = \begin{bmatrix} \mathbf{I}_m \\ \mathbf{t}_G \end{bmatrix}, \tag{7a}$$

$$\mathbf{t}_G = -\mathbf{K}_{ss}^{-1}\mathbf{K}_{ms}^{\mathrm{T}}, \tag{7b}$$

where subscript "G" represents the item of Guyan technique. The corresponding reduced stiffness and mass matrices are

$$\mathbf{K}_G = \mathbf{T}_G^{\mathrm{T}}\mathbf{K}\mathbf{T}_G = \mathbf{K}_{mm} - \mathbf{K}_{ms}\mathbf{K}_{ss}^{-1}\mathbf{K}_{ms}^{\mathrm{T}}, \tag{8a}$$

$$\mathbf{M}_G = \mathbf{T}_G^{\mathrm{T}}\mathbf{M}\mathbf{T}_G = \mathbf{M}_{mm} - \mathbf{M}_{ms}\mathbf{K}_{ss}^{-1}K_{ms}^{\mathrm{T}} - \mathbf{K}_{ms}\mathbf{K}_{ss}^{-1}\mathbf{M}_{ms}^{\mathrm{T}} + K_{ms}\mathbf{K}_{ss}^{-1}\mathbf{M}_{ss}\mathbf{K}_{ss}^{-1}\mathbf{K}_{ms}^{\mathrm{T}}. \tag{8b}$$

It is apparent that Guyan reduction is accurate at frequency of zero. As the frequency increases, the error becomes more significant.

According to Friswell et al. [10], the IRS technique is equivalent to approximately expanding the inverse matrix in Eq. (6b) by neglecting the higher orders of λ. This paper, however, begins with Eq. (6b) without any approximation. Pre-multiplying the dynamic matrix ($\mathbf{K}_{ss}$–$\lambda\mathbf{M}_{ss}$) in Eq. (6b), one has

$$(\mathbf{K}_{ss} - \lambda\mathbf{M}_{ss})\mathbf{t} = -(\mathbf{K}_{ms}^{\mathrm{T}} - \lambda\mathbf{M}_{ms}^{\mathrm{T}}) \tag{9}$$

and

$$\mathbf{K}_{ss}\mathbf{t} = -\mathbf{K}_{ms}^{\mathrm{T}} + \lambda(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}). \tag{10}$$

Then

$$\begin{aligned}\mathbf{t} &= -\mathbf{K}_{ss}^{-1}\mathbf{K}_{ms}^{\mathrm{T}} + \lambda\mathbf{K}_{ss}^{-1}(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}) \\ &= \mathbf{t}_G + \lambda\mathbf{K}_{ss}^{-1}(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}) \\ &= \mathbf{t}_G + \lambda\mathbf{t}_d\end{aligned} \tag{11a}$$

and correspondingly, the transformation matrix between the master d.o.f. and the full d.o.f.'s becomes

$$\mathbf{T} = \begin{bmatrix} \mathbf{I}_m \\ \mathbf{t}_G + \lambda\mathbf{t}_d \end{bmatrix} = \mathbf{T}_G + \lambda\begin{bmatrix} 0 \\ \mathbf{t}_d \end{bmatrix}, \tag{11b}$$

where

$$\begin{aligned}\mathbf{t}_d &= \mathbf{K}_{ss}^{-1}(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}) \\ &= \mathbf{K}_{ss}^{-1}\left[\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}(\mathbf{t}_G + \lambda\mathbf{t}_d)\right] \\ &= \mathbf{K}_{ss}^{-1}(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}_G) + \lambda\mathbf{K}_{ss}^{-1}\mathbf{M}_{ss}\mathbf{t}_d.\end{aligned} \tag{12}$$

One can find that transformation matrix $\mathbf{t}_d$ is frequency dependent in an iterative form, rather than a constant approximation. For example, when $\mathbf{t}_d = \mathbf{0}$ the present method equals to Guyan reduction. It also can be proved that the present method is similar to IRS technique as $\mathbf{t}_d = \mathbf{K}_{ss}^{-1}(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}_G)$. In order to derive the accurate transformation matrix $\mathbf{t}_d$, an iterative scheme is employed in present paper, as will be described later.

Substituting Eq. (11b) into $\mathbf{K}_R$=$\mathbf{T}^{\mathrm{T}}\mathbf{K}\mathbf{T}$, one can get

$$\begin{aligned}\mathbf{K}_R &= \left(\mathbf{T}_G + \lambda\begin{bmatrix}\mathbf{0}\\ \mathbf{t}_d\end{bmatrix}\right)^{\mathrm{T}}\mathbf{K}\left(\mathbf{T}_G + \lambda\begin{bmatrix}\mathbf{0}\\ \mathbf{t}_d\end{bmatrix}\right)\\ &= \mathbf{T}_G^{\mathrm{T}}\mathbf{K}\mathbf{T}_G + \lambda\begin{bmatrix}\mathbf{0} & \mathbf{t}_{\mathbf{d}}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\mathbf{K}_{mm} & \mathbf{K}_{ms}\\ \mathbf{K}_{ms}^{\mathrm{T}} & \mathbf{K}_{ss}\end{bmatrix}\begin{bmatrix}\mathbf{I}_m\\ \mathbf{t}_G\end{bmatrix} + \lambda\begin{bmatrix}\mathbf{I}_m & \mathbf{t}_G^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\mathbf{K}_{mm} & \mathbf{K}_{ms}\\ \mathbf{K}_{ms}^{\mathrm{T}} & \mathbf{K}_{ss}\end{bmatrix}\begin{bmatrix}\mathbf{0}\\ \mathbf{t}_d\end{bmatrix}\\ &\quad + \lambda^2\begin{bmatrix}\mathbf{0} & \mathbf{t}_d^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\mathbf{K}_{mm} & \mathbf{K}_{ms}\\ \mathbf{K}_{ms}^{\mathrm{T}} & \mathbf{K}_{ss}\end{bmatrix}\begin{bmatrix}\mathbf{0}\\ \mathbf{t}_d\end{bmatrix}\\ &= \mathbf{K}_G + \lambda\mathbf{t}_d^{\mathrm{T}}(\mathbf{K}_{mm}^{\mathrm{T}} + \mathbf{K}_{ss}\mathbf{t}_G) + \lambda(\mathbf{K}_{ms} + \mathbf{t}_G^{\mathrm{T}}\mathbf{K}_{ss})\mathbf{t}_d + \lambda^2\mathbf{t}_d^{\mathrm{T}}\mathbf{K}_{ss}\mathbf{t}_d.\end{aligned} \tag{13}$$

Nothing $\mathbf{K}_{ms}^{\mathrm{T}} + \mathbf{K}_{ss}\mathbf{t}_G = \mathbf{K}_{ms}^{\mathrm{T}} - \mathbf{K}_{ss}\mathbf{K}_{ss}^{-1}\mathbf{K}_{ms}^{\mathrm{T}} = \mathbf{0}$ and similarly $\mathbf{K}_{ms} + \mathbf{t}_G^{\mathrm{T}}\mathbf{K}_{ss} = \mathbf{0}$, above equation becomes

$$\mathbf{K}_R = \mathbf{K}_G + \lambda^2\mathbf{t}_d^{\mathrm{T}}\mathbf{K}_{ss}\mathbf{t}_d. \tag{14}$$

Similarly, we can obtain the reduced mass matrix as

$$\mathbf{M}_R = \mathbf{M}_G + \lambda\mathbf{t}_d^{\mathrm{T}}(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}_G) + \lambda(\mathbf{M}_{ms} + \mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss})\mathbf{t}_d + \lambda^2\mathbf{t}_d^{\mathrm{T}}\mathbf{M}_{ss}\mathbf{t}_d. \tag{15}$$

Therefore, the dynamic matrix of the reduced model is

$$\begin{aligned}\mathbf{Z}_R(\lambda) &= \mathbf{K}_R - \lambda\mathbf{M}_R\\ &= (\mathbf{K}_G + \lambda^2\mathbf{t}_d^{\mathrm{T}}\mathbf{K}_{ss}\mathbf{t}_d)\\ &\quad - \lambda\left[\mathbf{M}_G + \lambda\mathbf{t}_d^{\mathrm{T}}(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}_G) + \lambda(\mathbf{M}_{ms} + \mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss})\mathbf{t}_d + \lambda^2\mathbf{t}_d^{\mathrm{T}}\mathbf{M}_{ss}\mathbf{t}_d\right].\end{aligned} \tag{16}$$

In the right side of above equation, $\mathbf{t}_d$ of the second term is substituted by Eq. (12) which leads to

$$\begin{aligned}\lambda^2\mathbf{t}_d^{\mathrm{T}}\mathbf{K}_{ss}\mathbf{t}_d &= \lambda^2\mathbf{t}_d^{\mathrm{T}}\mathbf{K}_{ss}\left[\mathbf{K}_{ss}^{-1}(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}_G) + \lambda\mathbf{K}_{ss}^{-1}\mathbf{M}_{ss}\mathbf{t}_d\right]\\ &= \lambda^2\mathbf{t}_d^{\mathrm{T}}(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}_G) + \lambda^3\mathbf{t}_d^{\mathrm{T}}\mathbf{M}_{ss}\mathbf{t}_d.\end{aligned} \tag{17}$$

Substituting Eq. (17) into Eq. (16) and removing the identical items, one can get

$$\begin{aligned}\mathbf{Z}_R(\lambda) &= \mathbf{K}_G - \lambda\left[\mathbf{M}_G + \lambda(\mathbf{M}_{ms} + \mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss})\mathbf{t}_d\right]\\ &= \mathbf{K}_G - \lambda\mathbf{M}(\lambda).\end{aligned} \tag{18}$$

The stiffness matrix, $\mathbf{K}_G$ is the same as that of Guyan method in the form of Eq. (8a) but the mass matrix becomes that of the Guyan method with additional frequency-dependent perturbation. Therefore, the initial eigenvalue problem with order of N is transformed into a frequency-dependent eigenvalue problem with reduced order of m, i.e.,

$$[\mathbf{K}_G - \lambda\mathbf{M}(\lambda)]\mathbf{\Phi}_m = \mathbf{0}. \tag{19}$$

3. Solution of frequency-dependent eigenproblem

To derive the first n ($n<m$) eigenvalues and corresponding eigenvectors in Eq. (19), an eigensensitivity-based technique proposed by Lin and Lim [16], is employed here. It starts with the normal eigenvalue problem with constant property matrices. The frequency-dependent matrices can be treated as constant ones with additional perturbations. Then the eigenvalues and eigenvectors of the perturbed system are obtained based on sensitivity analysis. The process is simply introduced here for clearness.

A general eigenvalue problem with constant system matrices is established for the rth mode as

$$\left(\mathbf{K}^0 - \lambda_r^0\mathbf{M}^0\right)\mathbf{\Phi}_r^0 = \mathbf{0}. \tag{20}$$

Without loss of generality, a frequency-dependent eigenvalue problem is

$$[\mathbf{K}(\lambda) - \lambda_r\mathbf{M}(\lambda)]\mathbf{\Phi}_r = \mathbf{0}. \tag{21}$$

The system matrices and corresponding eigensolutions are expressed as the initial ones plus perturbations, i.e.,

$$\begin{aligned} \mathbf{K}(\lambda) &= \mathbf{K}^0 + \Delta\mathbf{K}, \\ \mathbf{M}(\lambda) &= \mathbf{M}^0 + \Delta\mathbf{M}, \\ \lambda_r &= \lambda_r^0 + \Delta\lambda_r, \\ \mathbf{\Phi}_r &= \mathbf{\Phi}_r^0 + \Delta\mathbf{\Phi}_r. \end{aligned} \tag{22}$$

Therefore, Eq. (21) can be rewritten as

$$\left[\left(\mathbf{K}^0 + \Delta\mathbf{K}\right) - \left(\lambda_r^0 + \Delta\lambda_r\right)\left(\mathbf{M}^0 + \Delta\mathbf{M}\right)\right]\mathbf{\Phi}_r = \mathbf{0}, \tag{23}$$

then

$$\left(\mathbf{K}^0 - \lambda_r^0\mathbf{M}^0\right)\mathbf{\Phi}_r + \left(\Delta\mathbf{K} - \lambda_r^0\Delta\mathbf{M}\right)\mathbf{\Phi}_r - \Delta\lambda_r\left(\mathbf{M}^0 + \Delta\mathbf{M}\right)\mathbf{\Phi}_r = \mathbf{0}. \tag{24}$$

Left-multiplying $\left(\mathbf{\Phi}_r^0\right)^{\mathrm{T}}$ and noting $\left(\mathbf{\Phi}_r^0\right)^{\mathrm{T}}\left(\mathbf{K}^0 - \lambda_r^0\mathbf{M}^0\right) = \mathbf{0}$, it has

$$\Delta\lambda_r = \frac{\left(\mathbf{\Phi}_r^0\right)^{\mathrm{T}}\left(\Delta\mathbf{K} - \lambda_r^0\Delta\mathbf{M}\right)\mathbf{\Phi}_r}{\left(\mathbf{\Phi}_r^0\right)^{\mathrm{T}}\left(\mathbf{M}^0 + \Delta\mathbf{M}\right)\mathbf{\Phi}_r}. \tag{25}$$

Then the rth eigenvalue is

$$\lambda_r = \lambda_r^0 + \Delta\lambda_r = \lambda_r^0 + \frac{\left(\mathbf{\Phi}_r^0\right)^{\mathrm{T}}\left(\Delta\mathbf{K} - \lambda_r^0\Delta\mathbf{M}\right)\mathbf{\Phi}_r}{\left(\mathbf{\Phi}_r^0\right)^{\mathrm{T}}\left(\mathbf{M}^0 + \Delta\mathbf{M}\right)\mathbf{\Phi}_r}. \tag{26}$$

To get the rth eigenvector, Eq. (21) can also be rewritten as

$$\left[\left(\mathbf{K}^0 + \Delta\mathbf{K}\right) - \lambda_r\left(\mathbf{M}^0 + \Delta\mathbf{M}\right)\right]\left(\mathbf{\Phi}_r^0 + \Delta\mathbf{\Phi}_r\right) = \mathbf{0}. \tag{27}$$

Then

$$\left(\mathbf{K}^0 - \lambda_r\mathbf{M}^0\right)\mathbf{\Phi}_r^0 + \left(\mathbf{K}^0 - \lambda_r\mathbf{M}^0\right)\Delta\mathbf{\Phi}_r + (\Delta\mathbf{K} - \lambda_r\Delta\mathbf{M})\mathbf{\Phi}_r = \mathbf{0} \tag{28}$$

and

$$\left(\mathbf{K}^0 - \lambda_r^0\mathbf{M}^0\right)\mathbf{\Phi}_r^0 - \Delta\lambda_r\mathbf{M}^0\mathbf{\Phi}_r^0 + \left(\mathbf{K}^0 - \lambda_r\mathbf{M}^0\right)\Delta\mathbf{\Phi}_r + (\Delta\mathbf{K} - \lambda_r\Delta\mathbf{M})\mathbf{\Phi}_r = \mathbf{0}. \tag{29}$$

It results in

$$\Delta\boldsymbol{\Phi}_r = \left(\mathbf{K}^0 - \lambda_r\mathbf{M}^0\right)^{-1}\left[(\lambda_r\Delta\mathbf{M} - \Delta\mathbf{K})\boldsymbol{\Phi}_r + \Delta\lambda_r\mathbf{M}^0\boldsymbol{\Phi}_r^0\right]. \tag{30}$$

It must be noted that since λ_r has been estimated already and different from λ_r^0, $\left(\mathbf{K}^0 - \lambda_r\mathbf{M}^0\right)$ is not singular and its inverse exists. From spectral decomposition, the inverse matrix in the above equation can be calculated as

$$\left(\mathbf{K}^0 - \lambda_r\mathbf{M}^0\right)^{-1} = \sum_{i=1}^{m}\frac{\boldsymbol{\Phi}_i^0\left(\boldsymbol{\Phi}_i^0\right)^{\mathrm{T}}}{\lambda_i^0 - \lambda_r}. \tag{31}$$

Upon substitution of Eq. (31), and noting the orthorgonality relation of the eigenvectors,

$$\sum_{i=1}^{m}\frac{\boldsymbol{\Phi}_i^0\left(\boldsymbol{\Phi}_i^0\right)^{\mathrm{T}}}{\lambda_i^0 - \lambda_r}\left(\Delta\lambda_r\mathbf{M}^0\boldsymbol{\Phi}_r^0\right) = \left(\lambda_r - \lambda_r^0\right)\sum_{i=1}^{m}\frac{\boldsymbol{\Phi}_i^0\left(\boldsymbol{\Phi}_i^0\right)^{\mathrm{T}}\mathbf{M}^0\boldsymbol{\Phi}_r^0}{\lambda_i^0 - \lambda_r} = -\boldsymbol{\Phi}_r^0, \tag{32}$$

Eq. (30) becomes

$$\Delta\boldsymbol{\Phi}_r = \sum_{i=1}^{m}\frac{\boldsymbol{\Phi}_i^0\left(\boldsymbol{\Phi}_i^0\right)^{\mathrm{T}}}{\lambda_{\mathrm{i}}^0 - \lambda_r}[(\lambda_r\Delta\mathbf{M} - \Delta\mathbf{K})\boldsymbol{\Phi}_r] - \boldsymbol{\Phi}_r^0. \tag{33}$$

Then the rth eigenvector is

$$\boldsymbol{\Phi}_r = \Delta\boldsymbol{\Phi}_r + \boldsymbol{\Phi}_r^0 = \sum_{i=1}^{m}\frac{\boldsymbol{\Phi}_i^0\left(\boldsymbol{\Phi}_i^0\right)^{\mathrm{T}}}{\lambda_i^0 - \lambda_i^0}[(\lambda_r\Delta\mathbf{M} - \Delta\mathbf{K})\boldsymbol{\Phi}_r]. \tag{34}$$

After λ_r and $\boldsymbol{\Phi}_r$ are estimated, the matrices $\Delta\mathbf{K}$ and $\Delta\mathbf{M}$ are updated according to the current eigenvalue and the new eigenpairs are solved again. An iterative strategy based on Eqs. (26) and (34) can be developed to obtain the eigensolutions of Eq. (21) for each mode. Since $\boldsymbol{\Phi}_r$ in Eqs. (26) and (34) is unknown in advance, the values in the previous iteration are used instead. After $\boldsymbol{\Phi}_r$ is obtained as Eq. (34), a better estimation of λ_r can be corrected by using Eq. (26) again. Numerical examples show that this estimation-correction process improves the results significantly.

Now we apply the iterative technique to the problem of this paper, formulated as Eq. (19). The property matrices and corresponding eigensolutions of Guyan method are taken as the initial ones. It is observed that

$$\mathbf{K}^0 = \mathbf{K}_G, \quad \Delta\mathbf{K} = 0, \tag{35a, b}$$

$$\mathbf{M}^0 = \mathbf{M}_G, \quad \Delta\mathbf{M} = \lambda\left(\mathbf{M}_{ms} + \mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss}\right)\mathbf{t}_d, \tag{36a, b}$$

where $\mathbf{K}_G$, $\mathbf{M}_G$ and to are constant with form of Eqs. (8a), (8b) and (7b), respectively, $\Delta\mathbf{M}$ depends on the eigenvalue and $\mathbf{t}d$. Based on Eq. (12), $\mathbf{t}_d$ can also be iteratively obtained as

$$\mathbf{t}_d^{(k)} = \mathbf{K}_{ss}^{-1}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}_G\right) + \lambda^{(k-1)}\mathbf{K}_{ss}^{-1}\mathbf{M}_{ss}\mathbf{t}_d^{(k-1)} \tag{37}$$

for $k = 1,2, \ldots$.

The complete analysis procedure is summarized as:

1. Partitioning the original matrices $\mathbf{K}$ and $\mathbf{M}$ into the sub-matrices $\mathbf{K}_{mm}$, $\mathbf{K}_{ms}$, $\mathbf{K}_{ss}$, $\mathbf{M}_{mm}$, $\mathbf{M}_{ms}$, $\mathbf{M}_{ss}$, according to the master d.o.f and slave d.o.f;

2. Calculating matrices $\mathbf{K}^0$ and $\mathbf{M}^0$, from which the eigenvalues and eigenvectors, λ^0 and $\mathbf{\Phi}^0$ can be obtained;
3. For the rth mode ($r = 1, 2, \ldots, m$), initializing $\lambda_r^{(0)} = \lambda_r^0$, $\mathbf{\Phi}_r^{(0)} = \mathbf{\Phi}_r^0$ and $\mathbf{t}_d^{(0)} = \mathbf{K}_{ss}^{-1}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}_G\right)$;
4. For the kth iteration ($k=1, 2, \ldots$), computing

$$(1)\ \mathbf{t}_d^{(k)} = \mathbf{K}_{ss}^{-1}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}_G\right) + \lambda_r^{(k-1)}\mathbf{K}_{ss}^{-1}\mathbf{M}_{ss}\mathbf{t}_d^{(k-1)},$$

$$(2)\ \Delta\mathbf{M}^{(k)} = \lambda_r^{(k-1)}\left(\mathbf{M}_{ms} + \mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss}\right)\mathbf{t}_d^{(k)},$$

$$(3)\ \mathbf{M}^{(k)} = \mathbf{M}^0 + \Delta\mathbf{M}^{(k)},$$

$$(4)\ \tilde{\lambda}_r^{(k)} = \lambda_r^0 - \frac{\lambda_r^0\left(\mathbf{\Phi}_r^0\right)^{\mathrm{T}}\Delta\mathbf{M}^{(k)}\mathbf{\Phi}_r^{(k-1)}}{\left(\mathbf{\Phi}_r^0\right)^{\mathrm{T}}\mathbf{M}^{(k)}\mathbf{\Phi}_r^{(k-1)}} = \lambda_r^0\frac{\left(\mathbf{\Phi}_r^0\right)^{\mathrm{T}}\mathbf{M}^0\mathbf{\Phi}_r^{(k-1)}}{\left(\mathbf{\Phi}_r^0\right)^{\mathrm{T}}\mathbf{M}^{(k)}\mathbf{\Phi}_r^{(k-1)}},$$

$$(5)\ \mathbf{\Phi}_r^{(k)} = \tilde{\lambda}_r^{(k)}\sum_{i=1}^{m}\frac{\mathbf{\Phi}_r^0\left(\mathbf{\Phi}_i^0\right)^{\mathrm{T}}}{\lambda_i^0 - \tilde{\lambda}_r^{(k)}}\Delta\mathbf{M}^{(k)}\mathbf{\Phi}_r^{(k-1)},$$

$$(6)\ \lambda_r^{(k)} = \lambda_r^0\frac{\left(\mathbf{\Phi}_r^0\right)^{\mathrm{T}}\mathbf{M}^0\mathbf{\Phi}_r^{(k)}}{\left(\mathbf{\Phi}_r^0\right)^{\mathrm{T}}\mathbf{M}^{(k)}\mathbf{\Phi}_r^{(k)}};$$

5. If $\left|\lambda_r^{(k)} - \lambda_r^{(k-1)}\right|/\lambda_r^{(k)} \leqslant \varepsilon$ then stop for the mode, otherwise set $k = k + 1$ and return to Step 4.

The iteration procedure is numerically accurate and computationally efficient as will be shown later. In the viewpoint of computation cost, the present method only involves solving eigenfunction once and computing some matrices as in Guyan reduction. Then for each mode, the eigenvalue is derived directly and only a small amount of extra computation is needed since most operations are matrix–vector multiplications and no matrix inverse or eigensolution is required.

4. Example 1: a two-level RC frame

Two structures are applied to illustrate the effectiveness and accuracy of the proposed algorithm. In each example, the results obtained by the proposed method are compared with those by Guyan reduction, IRS technique, iterated IRS [10] and the exact solutions obtained by using full FE models. It has been found that the selection of master d.o.f. certainly affects the validity frequency range [4,5,14]. However, this is not the focus of the present paper. In this paper, the master d.o.f.'s are uniformly distributed so that the frequency f_s are relatively large [3].

The first example is a 1:12 scaled two-story reinforced concrete (RC) frame model [17], as shown in Fig. 1. The cross-section area of the columns and the beams are 6.20×10^{-4} and 3.80×10^{-3} m^2, and the in-plane moment of inertia are 4.97×10^{-8} and 2.30×10^{-7} m^4, respectively. The mass density of the concrete is measured as 2.45×10^3 kg/m^3 and Young's modulus 21.5 GPa.The structure is modelled by 24 Euler–Bernoulli beam elements, as shown in

Fig. 1. Each node has three d.o.f. (horizontal, vertical displacements and rotation) and results in 66 d.o.f. in all. The first five natural frequencies are calculated with the full FE model and listed in Table 1. They are considered as the exact values for comparison purpose.

The lateral d.o.f. of 10 points, as shown in Fig. 1 are chosen as master d.o.f. The full model with order of 66 is reduced to that of 10. The first 5 natural frequencies obtained with different condensation techniques are compared in Table 1. It is noted that the cut-off frequency f_s is 206.740 Hz, slightly higher than the fifth frequency.

The table demonstrates that Guyan reduction does not reproduce the modal frequencies of the original system as expected. In particular, the first two modal frequencies by Guyan reduction

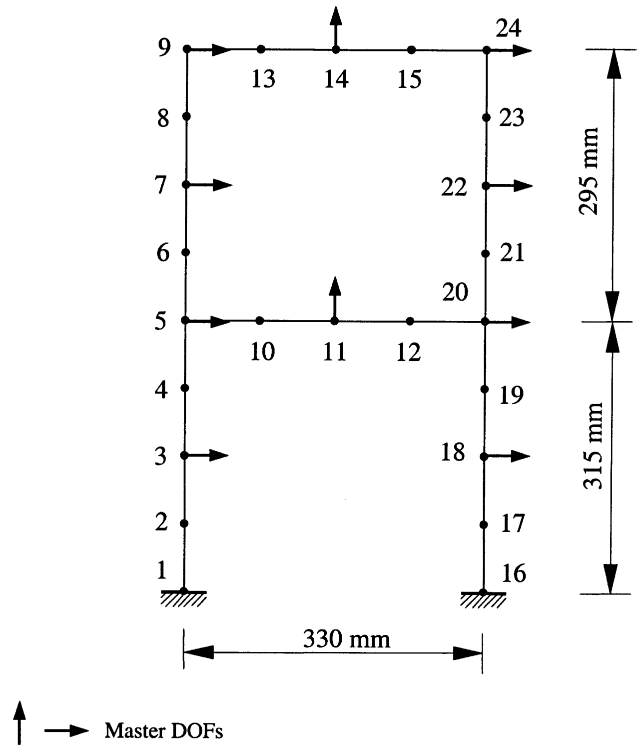

Fig. 1. Finite element model of the reinforced concrete frame and the master d.o.f.

Table 1
Comparison of the natural frequencies of the frame (Hz) (f_s = 206.740 Hz)

	Mode				
	1	2	3	4	5
Guyan reduction	13.477	40.552	98.883	113.170	267.514
IRS	13.476	40.537	96.892	111.602	259.409
Iterated IRS	13.476	40.537	96.880	111.594	209.495
Present method	13.476	40.537	96.880	111.594	206.907
Exact	13.476	40.537	96.880	111.594	206.705

method are acceptable for engineering application; the errors increase for the higher modes and become unacceptable for the modes whose actual frequencies are close to f_s. Similar results were observed in Ref. [4], that Guyan reduction method is acceptable in the frequency range of $[0, 0.3f_s]$. It also shows that IRS technique improves the results especially in the first four modal frequencies, but there is still a large error in the fifth mode. Using the iterated IRS [10], the fifth frequency approaches the exact value with the error of about 1.4%, after 20 iterations. With the present technique, as shown in the table, all the five frequencies obtained are very accurate. Even for the fifth one, the error is only about 0.1% after 20 iterations.

Table 2 shows the detailed iteration process for each modal frequency with the present algorithm. It demonstrates that the first four frequencies converge to the exact values within 5 iterations only but the fifth mode requires 20 iterations. This also verifies the importance of master selection. Nevertheless, the result is more accurate than that of the iterated IRS.

To investigate the accuracy of the eigenvectors, the full size eigenvectors are recovered as Eq. (4) after the eigenvectors of the master d.o.f.'s are obtained. The full size eigenvectors are compared with the exact ones and the error for mode r is evaluated as

$$e_r = \frac{|\Delta \mathbf{\Phi}_r|}{|\mathbf{\Phi}_r|} = \frac{\sqrt{\left(\mathbf{\Phi}_r^{exact} - \mathbf{\Phi}_r^{calculated}\right)^{\mathrm{T}}\left(\mathbf{\Phi}_r^{exact} - \mathbf{\Phi}_r^{calculated}\right)}}{\sqrt{\left(\mathbf{\Phi}_r^{exact}\right)^{\mathrm{T}}\mathbf{\Phi}_r^{exact}}}. \tag{38}$$

The results of Guyan reduction, IRS technique, iterated IRS [10] and present method are compared in Table 3.

It can be seen that, apart from the first two modes, the eigenvectors obtained with Guyan reduction have errors in excess of 9% and therefore not acceptable in most applications. IRS technique can get relatively accurate eigenvectors of the first four modes, but leads to totally wrong values for mode 5. Even with 20 iterations, the iterated IRS still has a large error in the fifth mode shape. With the present method, however, the errors of all modes of interest are very small. It is reminded that the first four modes converge within 5 iterations and only mode 5 involves 20

Table 2
Convergence of the natural frequencies with present technique (Hz)

Iteration no.	Mode				
	1	2	3	4	5
0	13.477	40.552	98.883	113.170	267.514
1	13.476	40.537	96.842	111.475	258.962
2		40.537	96.884	111.621	258.899
3			96.880	111.587	258.659
4				111.596	258.557
5				111.594	258.494
6					258.436
10					257.716
15					229.082
20					206.907
Exact	13.476	40.537	96.880	111.594	206.705

iterations. This table, along with Table 1, clearly shows that the eigensolutions of the frame can be accurately obtained with the present dynamic condensation algorithm.

It is noted that the eigenpairs are computed one by one in the present method rather than all required are calculated simultaneously as in other methods. Owing to the fact that the lowest eigenpairs converge very fast, the present method is effective when a few lowest eigenpairs are required as it is a case in practice. When more modes need to be computed, the efficiency will decrease. We are currently seeking to accelerate the present method by developing simultaneous method to save computation cost.

5. Example 2: the GARTEUR structure

The second example is the GARTEUR AG-11 structure [18], a grounded frame as shown in Fig. 2. Young's modulus is 7.5×10^{10} Pa and the mass density 2.80×10^{3} kg/m^3. The inertia of all members is 0.0756 m^4 and the cross-section area of the vertical, horizontal and diagonal bars are 0.006, 0.004 and 0.003 m^4, respectively. The structure is discretized by 78 Euler–Bernoulli beam elements, 74 nodes and 216 d.o.f.'s in total.

Twenty-one horizontal and vertical d.o.f., as shown in Fig. 2, are chosen as master d.o.f., i.e., the reduced model has order of 21. The first 10 natural frequencies obtained with different

Table 3
Error of the eigenvectors of the frame

	Mode (%)				
	1	2	3	4	5
Guyan reduction	0.04	0.25	11.67	9.26	336.47
IRS	0.00	0.00	1.06	0.71	322.76
Iterated IRS	0.00	0.00	0.02	0.01	51.21
Present method	0.00	0.00	0.00	0.01	0.03

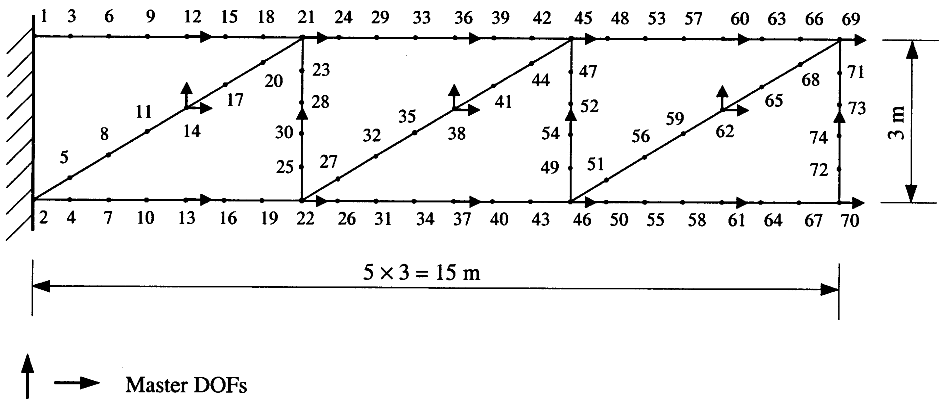

Fig. 2. Finite element model of the GARTEUR structure and the master d.o.f.

techniques are compared in Table 4 together with the averaged errors as compared with the exact values (the cut-off frequency f_s is 556.405 Hz). The errors of the calculated full size eigenvectors are computed based on Eq. (38) and are demonstrated in Table 5 with the averaged values.

It can be seen that the first two modal frequencies and eigenvectors obtained by Guyan reduction method are acceptable but for the higher modes the errors are significant. It also shows that IRS technique improves the results in the modal frequencies but cannot predict accurate eigenvectors for modes 6 – 10. With the iterated IRS and present technique after 5 iterations, the errors of all frequencies and eigenvectors are very small. The average of the errors shows that the results obtained by the present method are a little more accurate than those by the iterated IRS. This example again demonstrates that the present dynamic condensation algorithm can predict the eigensolutions of the structure effectively and accurately.

Table 4
Comparison of the natural frequencies of the GARTEUR structure (Hz) (f_s = 556.405 Hz)

Mode	Exact	Guyan	IRS	Iterated IRS	Present
1	45.150	45.208	45.150	45.150	45.150
2	79.047	79.174	79.047	79.047	79.047
3	227.182	230.781	227.187	227.182	227.182
4	249.669	256.867	249.674	249.669	249.668
5	363.564	373.651	363.640	363.565	363.566
6	437.724	479.977	438.469	437.738	437.742
7	445.902	490.693	446.534	445.903	445.886
8	469.207	520.167	470.171	469.237	469.192
9	488.879	536.698	489.709	488.914	488.895
10	511.074	565.609	511.917	511.103	511.098
Averaged error		5.85%	0.09%	2.29e-05	1.98e-05

Table 5
Error of the eigenvectors of the GARTEUR structure

Mode	Guyan (%)	IRS (%)	Iterated IRS (%)	Present (%)
1	0.16	0.00	0.00	0.00
2	0.38	0.00	0.00	0.00
3	7.60	0.18	0.02	0.00
4	7.43	0.18	0.02	0.00
5	10.06	1.02	0.09	0.02
6	20.03	3.84	0.35	0.25
7	19.86	2.92	0.08	0.02
8	22.36	4.55	0.58	0.14
9	23.27	6.53	0.82	0.44
10	28.04	7.15	1.09	1.46
Averaged error	13.92	2.64	0.31	0.23

6. Conclusions

A new effective eigensolution technique for large structures via iterated dynamic condensation has been proposed. This technique completely retains the inertia terms of the removed degrees of freedom, which yields a mass matrix frequency dependence in the reduced eigenfunction. The eigensolution is obtained by an iterative procedure combined with an eigensensitivity-based method.

The present algorithm has been applied to two practical examples. Numerical results have showed that the proposed technique accurately predicts all the frequencies and the eigenvectors in the cut-off frequency range. As compared with other commonly used condensation methods, Guyan reduction and IRS method, the proposed method is more accurate and is applicable to wider range of frequency of interest.

It has been found that the master selection is very important for convergence. Numerical examples show that the results converge very fast when a good master subset is selected, otherwise, convergence is very slow. The characteristics of convergence of the present method need to be studied further. Another drawback of the present method is that the eigenvalues are solved one mode by one mode, which reduces the efficiency when more modes are required. Simultaneous method is needed to save the computation cost.

References

[1] R.J. Guyan, Reduction of stiffness and mass matrices, American Institute of Aeronautics and Astronautics Journal 3 (2) (1965) 380.

[2] N. Bouhaddi, R. Fillod, A method for selecting master DOF in dynamic substructuring using the Guyan condensation method, Computers and Structures 45 (5) (1992) 941–946.

[3] N. Bouhaddi, R. Fillod, Model reduction by a simplified variant of dynamic condensation, Journal of Sound and Vibration 191 (2) (1996) 233–250.

[4] R.D. Henshell, J.H. Ong, Automatic masters for eigenvalue economisation, Earthquake Engineering and Structural Dynamics 3 (1975) 375–383.

[5] J.E.T. Penny, M.I. Friswell, S.D. Garvey, Automatic choice of measurement locations for dynamic testing, American Institute of Aeronautics and Astronautics Journal 32 (2) (1994) 407–414.

[6] M. Paz, Dynamic condensation method, American Institute of Aeronautics and Astronautics Journal 22 (5) (1984) 724–727.

[7] M. Paz, Modified dynamic condensation method, Journal of Structural Engineering, American Society of Chemical Engineers 115 (1) (1989) 234–238.

[8] J. O'Callahan, A procedure for an improved reduced system (IRS) model, Proceedings of Seventh International Modal Analysis Conference, 1989, pp. 17–21.

[9] P. Avitabile, J. O'Callahan, J. Milani, Comparison of system characteristics using various model reduction techniques, Proceedings of Seventh International Modal Analysis Conference, 1989, pp. 1109–1115.

[10] M.I. Friswell, S.D. Garvey, J.E.T. Penny, Model reduction using dynamic and iterated IRS techniques, Journal of Sound and Vibration 186 (2) (1995) 311–323.

[11] M.I. Friswell, S.D. Garvey, J.E.T. Penny, The convergence of the iterated IRS method, Journal of Sound and Vibration 211 (1) (1998) 123–132.

[12] L.E. Suarez, M.P. Singh, Dynamic condensation method for structural eigenvalue analysis, American Institute of Aeronautics and Astronautics Journal 30 (4) (1998) 1046–1054.

[13] Z.-Q. Qu, Z.-F. Fu, New structural dynamic condensation method for finite element models, American Institute of Aeronautics and Astronautics Journal 36 (7) (1998) 1320–1324.

[14] K.-O. Kim, M.-K. Kang, Convergence acceleration of iterative modal reduction methods, American Institute of Aeronautics and Astronautics Journal 39 (1) (2001) 134–140.
[15] A.Y.T. Leung, A simple dynamic substructure method, Earthquake Engineering and Structural Dynamics 16 (1998) 827–837.
[16] R.M. Lin, M.K. Lim, Complex eigensensitivity-based characterization of structures with viscoelastic damping, Journal of Acoustic Society of America 100 (5) (1996) 3182–3191.
[17] H. Hao, Y. Lu, Y. Xia, Statistical damage identification of structures with incomplete vibration data, Proceedings of Eighth International Conference on Structural Safety and Reliability, 2001.
[18] N.A.J. Lieven, D.J. Ewins, Expansion of modal data for correlation, Proceedings of Eighth International Modal Analysis Conference, 1990, pp. 605–609.